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On a Novel Geometric Representation of Rotation

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ABSTRACT

This paper presents a novel method of representing rotation and its application to representing the ranges of motion of coupled joints in the human body, using planar maps. The present work focuses on the viability of this representation for situations that relied on maps on a unit sphere. Maps on unit sphere have been used in diverse applications such as Gauss map, visibility maps, axis-angle and Euler-angle representations of rotation etc. Computations on spherical surface are difficult and computationally expensive; all the above applications suffer from problems associated with singularities at the poles. There are methods to represent the ranges of motion of such joints using two-dimensional spherical polygons. The present work proposes to use multiple planar domain "cube" in stead of a single spherical domain, to achieve the above objective. The parameterization on the planar domains is easy to obtain and convert to spherical coordinates. Further, there is no localized and extreme distortion of the parameter space and it gives robustness to the computations. The representation has been compared with the spherical representation in terms of computational ease and issues related to singularities. Methods have been proposed to represent joint range of motion and coupled degrees of freedom for various joints in digital human models (such as shoulder, wrist and fingers). A novel method has been proposed to represent twist in addition to the existing swing-swivel representation.

INTRODUCTION

Representation of rotation is an important issue in mechanics and robotics. Hence it also occurs naturally in the kinematic modeling of digital human models. Although the fundamental physical nature of rotation is well understood, its mathematical representation is non-trivial. There are many schemes of representing rotation in literature such as quaternion, Euler angles, Rodrigues parameters, roll-pitch-yaw, angle-axis etc. The primary characteristics in quaternion representation is unified representation of translation and rotation and that of angle-axis representation is its direct similarity with the operation being represented and the other representations view any rotation being equivalent to a sequence of canonical rotations. All the above representations can be mathematically expressed using vector-matrix formalism. Hence it provides a well studied framework of representation and manipulation of rotation. However, this very representational convenience brings with it the problems associated with non-commutativeness and singularity. The fundamental aspect in angle axis formalism is representation of the axis which is the problem of identification of a direction which is represented as a unit vector and an associated scalar, the angle of rotation. Thus representation of direction is closely related to the problem of representation of rotation. In homogeneous coordinate system, a direction is represented with the homogeneous component zero and hence it implies a point at infinity in projective geometry.

Representing directions in space is important for diverse applications; it occurs naturally in geography [1], astronomy and in the problems of Gauss maps [2], visibility map [3] [4]. In these applications a geometric rather than an algebraic representation of direction is more relevant. A point on the surface of a unit sphere thus becomes a natural choice for geometric representation of direction. The applications of this unit sphere representation of direction or motion axis include construction of spherical polygons [5] [6] for representing arbitrary ranges of motion of spherical joints and blocked directions in assembly sequence planning [7].

A sphere represented using an implicit equation is simple and convenient for classifying points in space to be inside, outside or on the sphere; it is not amenable for classification of different categories of points on the sphere itself. The problem appears when one needs to figure out whether or not a given direction of interest belongs to a given a set of feasible directions. In the context of digital human modeling, after computing the orientation of a link in a spherical joint, we need to verify whether or not it is within the joints range of motion. This necessitates parametric representation of sphere such that representation and classification of points on a sphere can, in principle be done in the parameter space itself [5]. This is equivalent to the usage of a two dimensional manifold for representing a point on a sphere. The latitude-longitude representation is more or less a natural choice in geography for the study of the consequence of the spin (causing day and night) and the tilt (causing seasonal variations). The conventional azimuth-elevation scheme in astronomy is also a natural

choice where the consequence of singularity at the poles have been effectively utilized for defining a universal coordinate frame through the identification of the pole-star and the direction of north. There is no such natural reason to use it in many engineering applications including multi-d.o.f. joint representation such as a universal or a spherical joint. This parametric representation is topologically equivalent to a *torus* of which the sphere is a *degenerate* case. The singularities in this case arise due to the method of parameterization itself and hence are representational singularities. However, all parameterization of a complete sphere would necessarily have singularities, a topological fact that can be derived from the celebrated Hairy Ball theorem [8]. On the other hand, a torus, though singularity free, is not a suitable object to support the geometric representation of direction because a ray from its centre could intersect it at two points or not intersect it at all. Therefore, a singularity free parameterization of a sphere can only be achieved through partitioning the sphere into disjoint domains. Many geometric modeling kernels, such as *SHAPES®*[1] have in fact adopted a dual hemisphere parameterization for a complete sphere to circumvent the problem of singularity. Obviously, this minimal domain sub-division does not have any planar counterpart. Tetrahedron and other homogeneous *triangulation* for parameterizing a sphere using multi planar domains have been mentioned in [1]. The celebrated "*bucky ball*" (truncated icosahederon) architecture for spherical domes actually sub-divides the spherical domain in terms of a minimum number of regular *pentagons* and *hexagons*. Any convex polygonal domain admits a singularity free parameterization such as using barycentric coordinates [9] or Cartesian coordinates over trimmed domains. Recently definition of barycentric coordinates for points in a spherical polygon has also been proposed [10]. We observe that a rectangular domain gives a natural parameterization of a planar domain (figure 1). All multi-domain parameterization of sphere would involve identification of a domain and a point on it for a given point on the sphere which is equivalent to a given direction of interest. A point on a convex object containing the origin represents a direction from the origin and passing through the given point. Hence for a given direction of interest (may be in homogeneous coordinates of a point), identification of a domain and an ordered set of parameters is a fundamental problem in dealing using singularity free multi-domain parametric representation of direction. In this paper we propose and study the use of a unit cube in stead of a unit sphere for uniquely identifying directions of interest, representation of joint ranges of motion and twist motion.

CONCEPT

Keeping in mind the importance of parameterization of directions and also the methods of such parameterizations which are devoid of singularities, we propose a special multi-planar domain, "*cube*" to parameterize and represent directions on space. In this section, we define the parameterization scheme on a cube and study the properties of the projections of the basic geometric entities on a parametric cube.

A cube is a platonic solid with six *planar* faces, twelve edges and eight vertices; all the adjacent faces are *orthogonal* to each other. These properties of planarity, orthogonality and symmetry make the cube the focus of this work.

DEFINITION AND PARAMETERIZATION - A parametric cube is defined as a cube with a specified length such that the geometric centre of the cube is coincident with the origin of the coordinate system and the faces are parallel to the coordinate planes as shown in figure 3(a). Since a cube is a symmetric object, each face is parameterized identically by constructing a local coordinate frame with its origin at the centre of the face and the axes parallel to the edges of the face (figure 3(b)). For case of a unit cube, the range of parameters is selected as ± 0.5 such that the global coordinates of a point on a face can be almost directly used for its parameters.

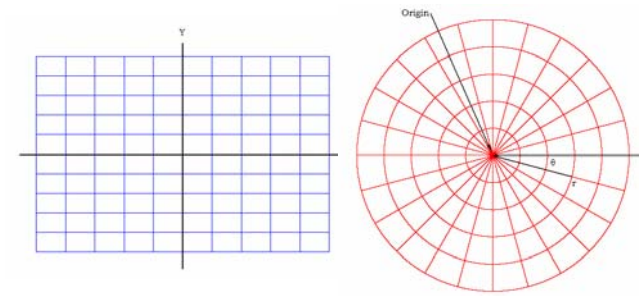

Fig 1: Rectangular Domain and Circular Domain

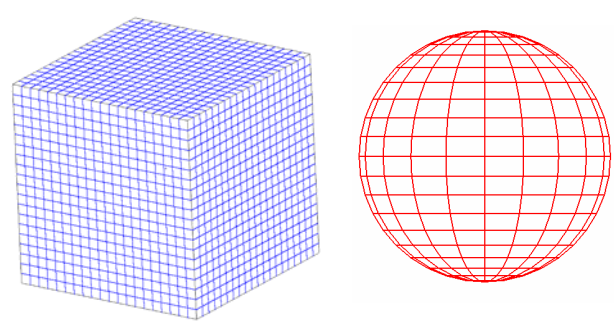

Fig 2: Parametric Cube and Parametric Sphere

REPRESENTING A DIRECTION ON A CUBE - Generally three standard representation schemes are followed for communicating the position of a point in three-dimensional space. They are namely, Cartesian (x, y, z), Spherical (R, θ, φ) and Cylindrical (R, θ, z) Schemes. Each of these can be used for unambiguous identification of a direction in space.

[1] A product of erstwhile XOX Corporation Minneapolis, MN, USA used in many geosciences application

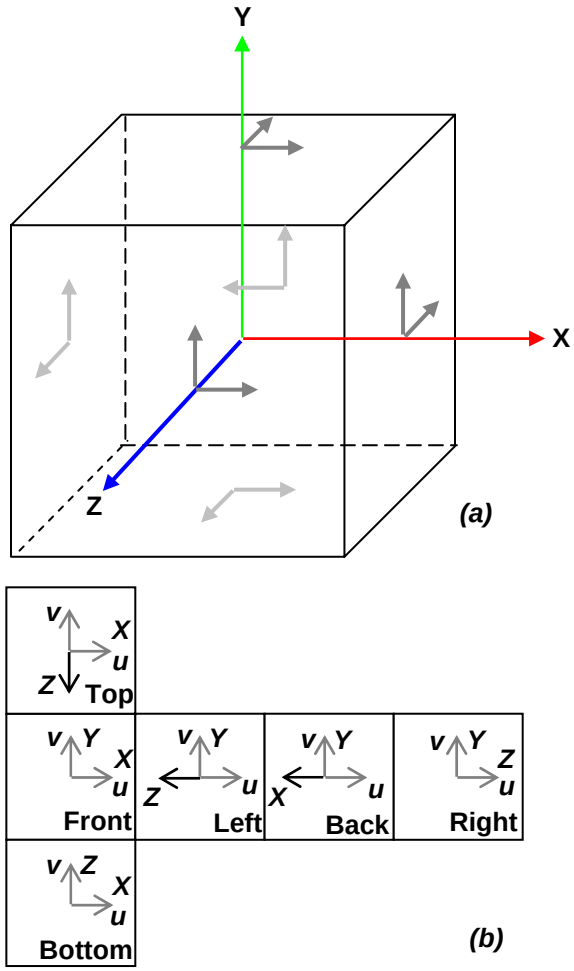

Fig 3: (a) Definition of a parametric cube,
(b) Parametric space of a opened cube

In conventional geometric representation of direction a spherical coordinate system is used and each direction is identified with a point on a unit sphere. From purely geometric point of view, any other convex object around the origin would also serve the same purpose since every ray from origin would intersect it at a well defined point on it. Thus the problem of parameterization of direction involves identification of points on the surface of the object with a unique set of associated parameters.

In the present context, we consider a set of all axis oriented cubes with their geometric centers at the origin. Thus, a 3D vector can be identified with an ordered triplet *(u, v, i)*, where '*u*' and 'v' are the coordinates of the point of intersection of the vector or its extension on the face *i* (figure 4). Intersections on the edges or vertices of the cube can be taken care of by following a convention, say on the orientation of the edges.

PROJECTING GEOMETRIC ENTITIES ON THE CUBE - The projection of geometric object on the cube is referred to as the *shadow* of the object in this paper. In this section, we study the shadows of a *point* and a *line-segment* on a cube.

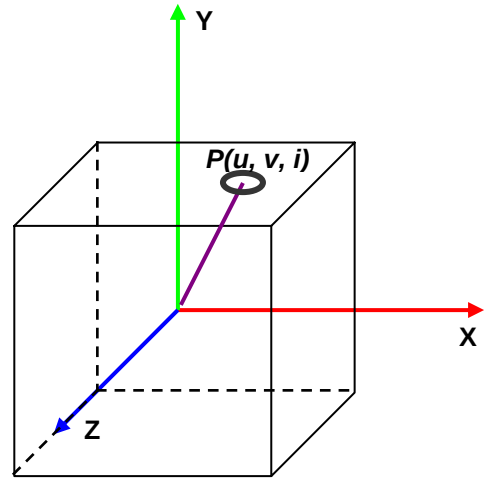

Fig 4: Representation of vector on cube

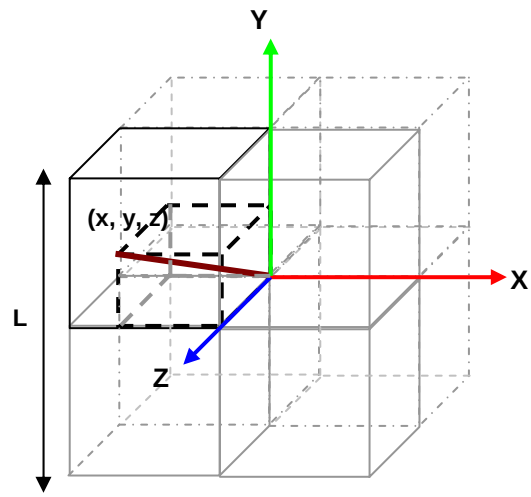

Fig 5: Unique Cube associated with a point
L = 2max{|x|, |y|, |z|}

<u>Lemma 1</u> - Every point or vector $V(x, y, z)$ can be associated with a unique cube $C_{\min}$ with its geometric centre at $O(0,0,0)$ such that *V* lies on the surface of the cube and the length a side of the cube is given by $2\max\{|x|,|y|,|z|\}$.

Proof: Since the position and orientation of the cube is predefined, we need only to verify the uniqueness of its dimension. Consider only the octant containing *V*. Now referring to figure 5, following observations can be made.

1. The point containment in the half-spaces defined by its faces is determined by independently considering a coordinate and the respective face.
2. If a half-space contains a given point, the half-space defined by a surface parallel to it but further away from the origin will also contain it.
3. Thus, the nearest axis-oriented plane containing the point will be defined by the respective

coordinates themselves. That is nearest plane parallel to x-plane is defined by the x-coordinate of *V*, and so on.

4. A point is contained in a parallelepiped if it is contained in each of the half-space defined by its faces. Thus, the smallest axis-oriented parallelepiped containing *V* and O has the line *OV* as its solid diagonal.
5. The smallest axis oriented cube that contains the parallelepiped is the one that contains its farthest face. Thus the size of the cube is max($|x|,|y|,|z|$).
6. The above cube is in only one octant with its one vertex at the origin. Hence the smallest cube with its center at the origin and containing the above cube is double its size.

■

$C_{\min}$ is referred to here as the *minimal cube* for *V* because it is the smallest cube to contain *V*. It is easy to see that *V* lies on the face corresponding to its largest coordinate in its own octant. This is referred to here as the *active face* of the cube induced by *V*.

Lemma 2 – If the faces of a cube were labeled in some order and then the *label* of the active face for the shadow of a point is *scale invariant*.

Proof: It is required to show that the label of the face intersected by an infinite ray from its center is independent of the size of the cube (figure 6).

Consider the cone defined the active face and the center of the cube induced by the point. Scaling is a linear operation with the collineation property that linear entities remain linear after the scaling operation. Hence, the ray from the center through the given point is scale invariant. A vertex of the cube lies on the same line joining center and itself before and after scaling. Thus, the cone is scale invariant. Thus the conical partitions of the space defined by the cube before scaling are same as the one after scaling. The plane of the face defining the cone also remains plane after scaling. Thus, a point originally lying on the face of the cube remains on the face of the cube after scaling. Thus the association of a point and the active face on the cube induced by it is persistent under the scaling operation. Hence the label of the active face for the shadow of a point is scale invariant.

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Theorem 1 – Every parametric cube admits a scale invariant parameterization of a given direction.

Proof: Let us take a direction in space and consider its shadows $P_{f1}(x_{f1}, y_{f1})$ and $P_{f2}(x_{f2}, y_{f2})$ on two cubes C_1 and C_2 as shown in figure 7(a).

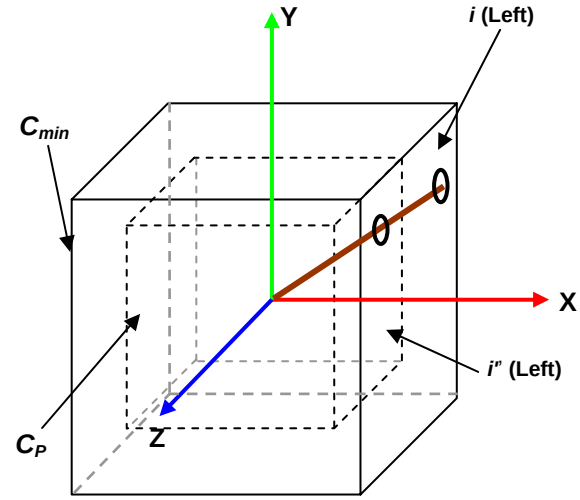

Fig 6: Same faces for original and projected points on cube

Here, (x_{f1}, y_{f1}) and (x_{f2}, y_{f2}) are the coordinates of these projections on C_1 and C_2 respectively on the face *f* (from lemma 2 we already know that the faces will be invariant irrespective of the cube considered). From figure 7, it can be observed that $\Delta OO_{f1}P_{f1}$ and $\Delta OO_{f2}P_{f2}$ are similar triangles giving the following relation:

$$\frac{x_{f1}}{L_1} = \frac{x_{f2}}{L_2} = u(say)$$

$$\frac{y_{f1}}{L_1} = \frac{y_{f2}}{L_2} = v(say)$$

The following observations are made from the equation above:

1. The values '*u*' and '*v*' are independent of the dimension of the cube on which the shadow of the direction is taken hence proving the theorem. The division of the coordinates of the projections can be considered equivalent to finding the shadow of the given direction on a unit cube.

2. If a cube of unit length is chosen for projecting the given direction, then '*u*' and '*v*' would lie within the range of ±0.5 and can be directly used as the parametric coordinates of the given direction. In other words, we choose a parameterization scheme for the cube which incorporates the property of scale invariance which we prove in the present theorem.

■

Thus, without loss of generality, the size of an edge of the cube can be selected as unity. Then the variable coordinates on an active face of parametric cube, directly give the parameters of the shadow of a direction on that face, provided axes of the parameters are appropriately identified (figure 3(b)).

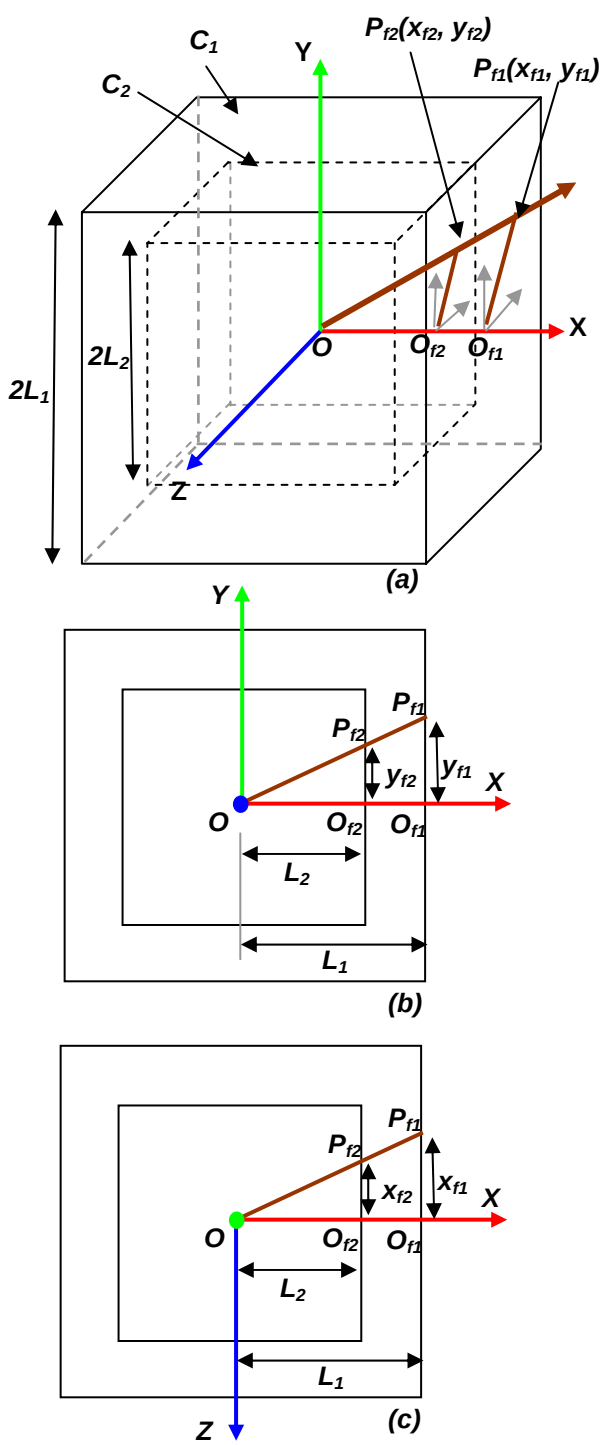

Fig 7: Scale invariance of a direction:
(a) 3D View, (b) Front View, (c) Top View

We can observe that the properties of planarity and convexity of faces offer many simple *parameterizations* of points in the faces (including their barycentric coordinates) in comparison to the case of a spherical surface (figure 2). In addition, a cube gives a desirable identical parameterization for all the domains due to symmetry. The use of a unit cube with axis-oriented faces and centre coinciding with the origin makes the task of *domain identification*, for a given direction, straightforward.

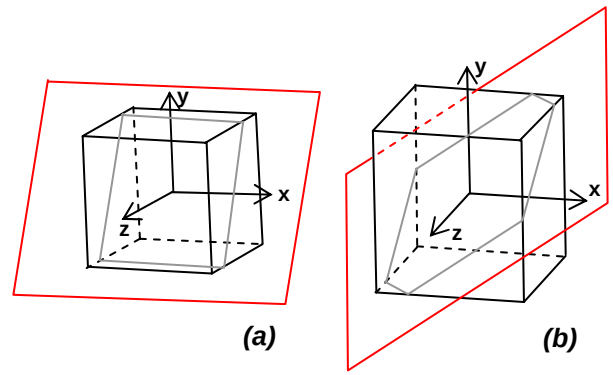

Fig 8: (a) Plane intersecting four edges of cube
(b) Plane intersecting six edges of cube

Lemma 3 - A plane passing through the centre of a cube intersects either four or six of its edges.

Proof: Since cube is a symmetric object, any plane passing through its centre (*in the present case since the centre is at the origin, planes passing through the origin are considered*) will divide the cube into two equal halves with four vertices on each side of the plane.

There can be two possibilities for such a division. Either the four vertices on either side lie on the same face or they don't.

If the sets four vertices on either side lie on the same face, the edges formed by them will intersect the plane outside the cube.

This implies that eight out of the twelve edges of the cube will intersect the plane outside the cube leaving only four edges intersecting it at points lying on the cube (figure 8(a)). On the other hand, if each set of four vertices does not form a face on the cube, we can two diagonally opposite vertices on the cube (one on each side of the plane) such that we get sic mutually perpendicular edges which intersect the plane externally (three on each side of the plane). Hence the remaining six faces will intersect the cube at points lying on the cube (figure 8(b)). It is to be noted here that situations where the plane contains the edges or vertices of the cube are not in consideration.

Lemma 4 - If a line lies on plane passing through the centre of a cube, its projection on the cube is divided in at most half the number of edges of the cube intersected by the plane.

Proof: Invoking Lemma 3, it is known that the number of edges that the plane passing through a cube's centre is either four or six. It is to be noted that by virtue of symmetry, in either cases the points of intersection will be placed symmetrically about the centre of the cube. Hence, if we construct two line-segments each joining the centre to one of the end-points of the given line, at most half of the points of intersection will lie in the region between these line-segments.

Further, if the number of intersection points is four, either none, one or two points at most can lie within the region between the constructed line-segments (figures 9(a), (b) and (c)).

On the other hand, if the number of intersection points is six, either none, one, two or three points at most can lie within the region between the constructed line-segments (figures 9(d), (e), (f) and (g)).

Conjecture 1 – If the projections of the end-points of a line on a cube lie on the same face, the projection of the line on the cube lies on the same face.

Since each face of a cube is a convex polygon, the conjecture follows from the definition of convexity (figures 10(a) and 11(a)).

Conjecture 2 - When the shadows of the end-points of a line on a cube lie on the adjacent faces the shadow of the line itself contains two segments if the plane containing the line and the centre of the cube intersects common edge of the adjacent faces. Otherwise, the shadow contains three segments.

Let us construct a plane which contains the given line P_1P_2 and the centre O of the cube. Let us also construct two line-segments OP_1 and OP_2.

If the plane intersects the common edge of the adjacent faces, then only one point can lie within the region between OP_1 and OP_2 (figures 9(b) and 9(e)). This is because there can be only one common edge for two adjacent faces. Hence the number of segments is two (figures 10(b) and 11(b)).

The case when the plane does not intersect the common edge is not possible when the plane intersects four edges of the cube since all the intersected faces which are adjacent should necessarily have a common edge which is intersected by the plane. Hence the only possibility left is of having six edges of intersection from Lemma 3.

Further, the plane must intersect at least one face associated with the edge which is intersected. Since the common face is not intersected, the only possibility is that the plane will intersect two edges which are the common edges between the adjacent faces and the face mutually perpendicular. Hence two points will lie within the region between OP_1 and OP_2. Hence the number of segments is three (figure 11(c)).

Conjecture 3 – When the shadows of the end-points of a line on a cube lie on the opposite faces the shadow of the line itself contains three segments if the plane containing the line and the centre of the cube intersects four edges of the cube. Otherwise, the shadow contains four segments.

If we construct a plane which contains the given line and the centre of the cube, then by using Lemma 3 we know that this plane can intersect either four or six edges of the cube.

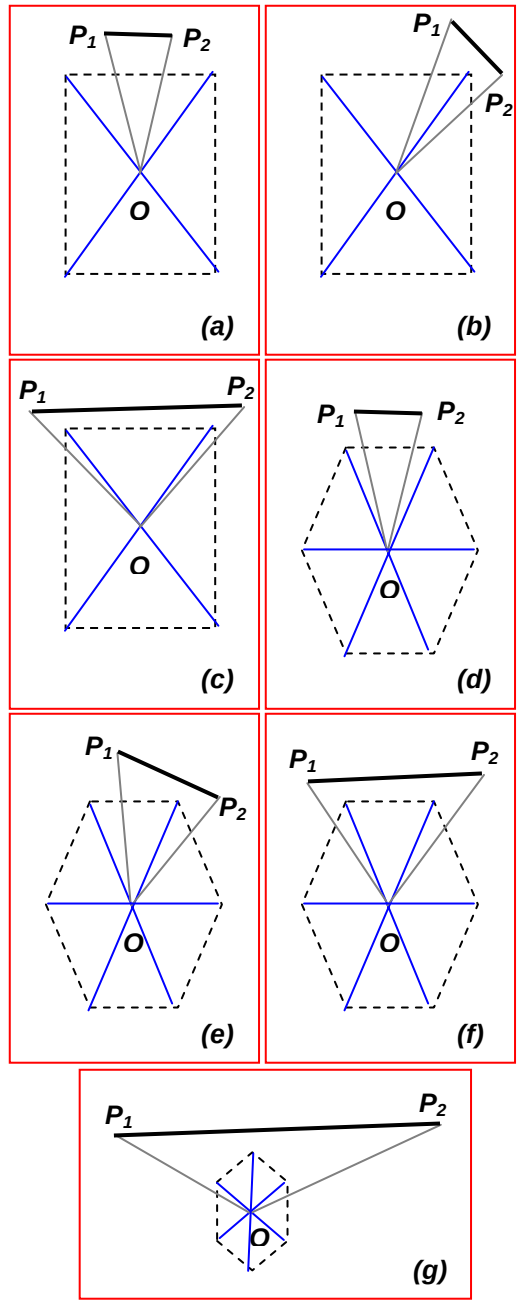

Fig 9: Number of intersections within the region of OP_1 and OP_2 on the plane containing O and P_1P_2

If the plane intersects four edges, there must be one face adjacent to both of these faces thus giving two common edges which are intersected by the plane. Hence only two points can lie within the region between OP_1 and OP_2 since (figure 9(c)). Hence the number of segments is three (figure 10(c)).

If the plane intersects six edges, there must be two faces adjacent to both of these faces thus giving two common edges which are intersected by the plane.

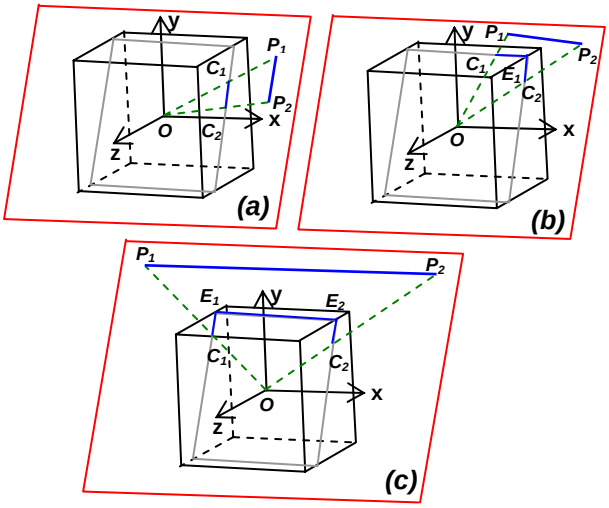

Fig 10: Four Edge Intersection, (a) Same face: single line, (b) Adjacent faces: 2 segments, (c) Opposite faces: 3 segments

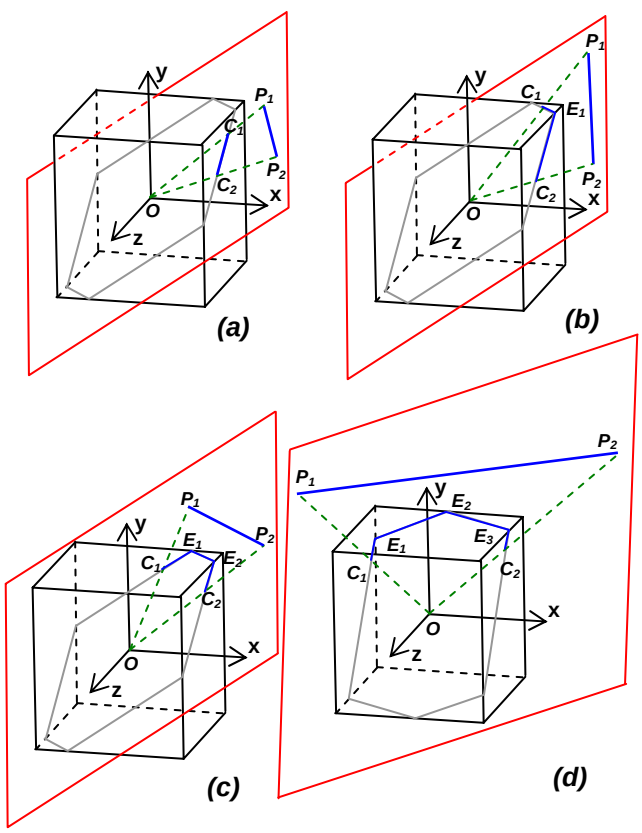

Fig 11: Six Edge Intersection, (a) Same face: single line, (b) Adjacent faces: 2 segments, (c) Adjacent faces: 3 segments (d) Opposite faces: 4 segments

Hence only three points can lie within the region between OP_1 and OP_2 since (figure 9(g)). Hence the number of segments is four (figure 11(d)).

Using Conjectures 1, 2 and 3, we can find the projections of a line P_1P_2 by finding the projection polygon of the plane formed by ΔOP_1P_2 on a parametric cube with its centre at the origin O. If C_1 and C_2 be the respective projections of P_1 and P_2 on a parametric cube, we can find the points of intersection of the plane such which lie within OP_1 and OP_2, and obtain the segments of the shadow of the line P_1P_2 (figures 10 and 11).

RESULTS

Keeping in view, the aim of classifying a direction with respect to a given set of directions, certain cases were simulated for both cuboid as well as spherical parameterization schemes for establishing methods and the results so obtained were compared. This was carried out using an interface developed for representing directions using cube and sphere parameterization schemes. In the first half of this section, we address the problem of direction classification by taking a triangle to represent a set of directions and a point in space to represent the direction which is to be classified with respect to this given set of directions. In the rest of this section we extend this idea by replacing a triangle with triangulated spherical surfaces and propose this extension for representing the ranges of motion of coupled joints. Lastly, we propose a method to incorporate the ranges of twist of arbitrary spherical joints.

COMPARISON OF CUBOID AND SPHERICAL REPRESENTATIONS - A triangle $\Delta P_1P_2P_3$ and a point vector P were taken in three-dimensional space and projected on a parametric cube and sphere. This triangle represents a cone in space containing all valid directions of concern and the point refers to a direction in space need to be classified with respect to this cone. Using shadows of the triangle and the point thus obtained in the parametric spaces of the cube and the sphere converts the classify-direction-in-cone problem to a point-in-loop problem. The parametric spaces of the cube and the sphere are then observed in context to two properties. One is the linearity of curves in the parametric space and the other is the quality and ease of point classification.

By point classification we mean the ability to tell whether the direction of the vector from the origin to the point of concern lies inside a given set of such directions. Practically this implies checking the position of the projection of the given point with respect to the closed area in the parametric space depicting the set of directions. Table 1 shows four distinct cases with different triangles at different positions and orientations and the respective points of concern. The corresponding figures (12, 13, 14 and 15) show the parametric spaces of the cube and sphere with the triangle and the point projected on it.

Table 1: Case A (Equilateral triangle containing pole placed far from sphere), Case B (Equilateral triangle containing pole placed near to sphere), Case C (Triangle with direction lying inside its cone), Case D (Triangle with direction lying inside its cone)

Case A	x	y	z
P1	0.5	4	0
P2	0.25	4	0.433
P3	0.25	4	-0.433
P	-0.27	4	0.2
Translation	0	0	0
Rotation	0	0	0
Case B	x	y	z
P1	2	1.5	0
P2	-1	1.5	1.732
P3	-1	1.5	-1.732
P	1.9	2	0.5
Translation	0	0	0
Rotation	0	0	0
Case C	x	y	z
P1	-2.05613	2	-0.40645
P2	-0.53973	2	2.7188
P3	4	2	-0.433
P	1.47299	4	2.0243
Translation	2.11542	0	0.916535
Rotation	0.225619	0	0
Case D	x	y	z
P1	0.204217	2	1.5306
P2	1.10307	2	-0.5
P3	3.55729	-1.66698	-0.8
P	1.64	1.2	1.28
Translation	0	0	0
Rotation	0	0	0

Cases A and B have been designed to show the effect of representational singularity in the parametric space. The triangle considered in these cases is an equilateral triangle on a plane parallel to the x-z plane placed such that its projection on the sphere contains the upper pole of the sphere. This ensures that the projection of one of the sides of the triangle crosses the longitude on the sphere with zero angle of azimuth. The line is thus broken in the parametric space owing to the constraint of the angular limits between 0 and 2π. In case A it is observed that as the triangle gets smaller or farther along the y-axis, the projection comes closer to the pole asymptotically giving a straight line in the parametric space. In either case the classification of a point with respect to the projected triangle becomes irrelevant since either there is no triangle or it is infinitely away. On the other hand, in cube representation, the triangle is simple scaled down without any distortion in shape. Hence the classification of the point becomes simple. Case B shows the parametric space of a triangle whose projection includes the pole but the triangle is neither too small, nor quite far away. In this case the classification of the point becomes difficult.

Through cases C and D, the issue of the non-linearity of spherical parametric space has been addressed. It is observed that spherical parameterization is non-linear in the sense that any three collinear points in space when projected on a sphere do not remain collinear in its parametric space. Hence, the triangle when viewed in the parametric space is a union of three curved lines. This gives rise to taking an additional decision as regards the computational accuracy up to which the lines of the triangle are to be projected (by taking segments of each line and projecting them). The classification of points which are close to the boundary of the triangle is difficult in this scenario. More the accuracy required, more is the computational expense. This problem remains non-existent in case of cube since all projections retain their linearity on the cube. Figure 14 shows how a point which is inside the triangle is easily classified using cube representation whereas it is classified differently in the spherical counterpart depending on the choice of level of accuracy (number of segments of each side of the triangle).

Figure 15 shows the same effect for a point which is outside the triangle but is shown to be inside if sufficient care is not taken for deciding the level of accuracy.

These effects magnify when the context is of smooth curves i.e. rational B-splines and other parametric curves. Methods to parameterize smooth curves using quaternion-approach have been discussed in [11]. In case of spherical projection and parameterization, the projected control polygons are not linearly related in parametric space. Hence the construction of the smooth curve even more difficult as the transformation of these curves from geometric space to parametric space involves transcendental expressions (inverse trigonometric functions).

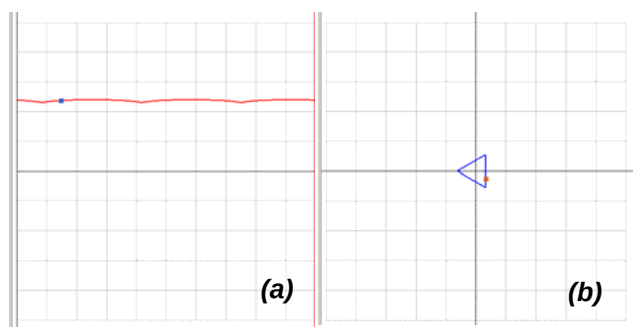

Fig 12: Case A, Triangle in (a) Spherical Parametric Space, (b) Cube Parametric Space

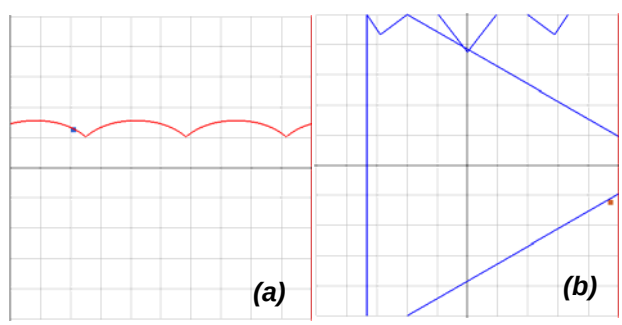

Fig 13: Case B, Triangle in (a) Spherical Parametric Space, (b) Cube Parametric Space

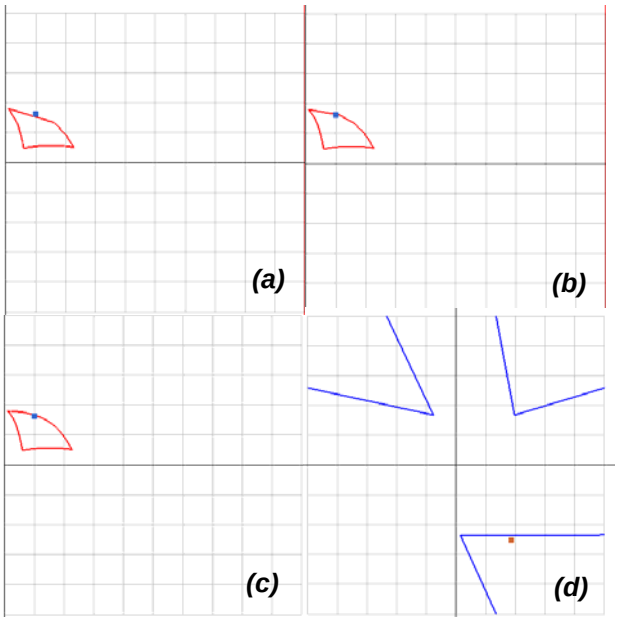

Fig 14: Case C, (a) 4 segments – Point Outside, (b) 8 segments – Point on boundary, (c) 100 segments – Point Inside, (d) Cube Parametric Space – Point Inside

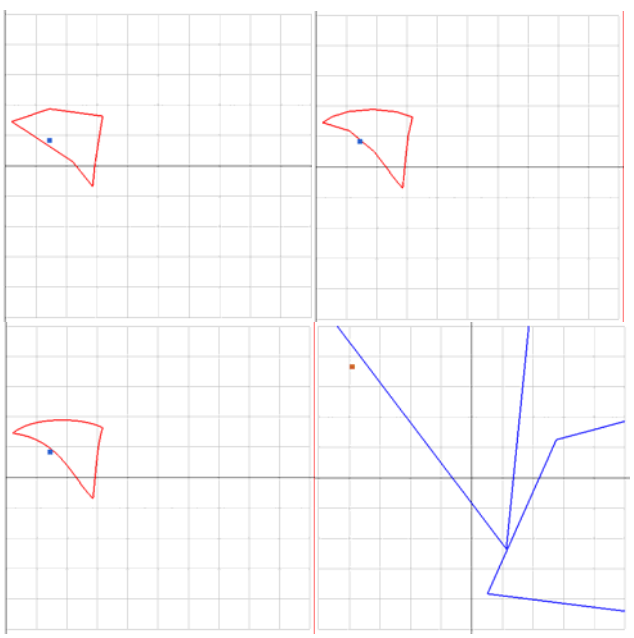

Fig 15: Case C, (a) 2 segments – Point Inside (b) 5 segments – Point on boundary (c) 100 segments – Point Outside (d) Cube Parametric Space – Point Inside

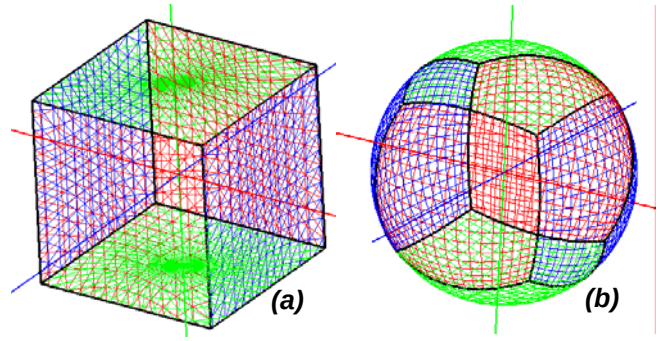

Fig 16: 3D View of (a) a parametric sphere projected on a cube, (b) a parametric cube projected on a sphere

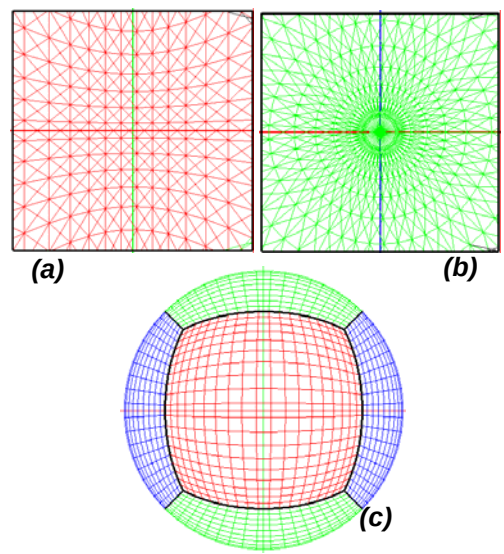

Fig 17: 2D View (a) Front view and (b) Top View of a parametric sphere project on a cube, (c) Front view of a parametric cube project on a sphere

On the other hand since all rational B-splines are projective invariant on a plane, the projections of the control points can be directly used to construct the projection of the spline. This fact can be used in the present context of representing joint ranges of motion for spherical and other coupled joints since the ranges of motion would be a set of directions which can be mapped to a surface encompassed by a closed and possibly continuous curve in space. In such situations, using spherical representation would be theoretically and computationally tedious and the use of cube for representation would be advantageous.

To appreciate the effect of the parametric spaces, a parametric sphere was projected on to a cube and that of a parametric cube on a sphere (figure16). It is clear that on the vertical faces, the points of the sphere are scattered non-uniformly showing the non-linearity of the spherical parametric space (figure 17(a)). Further, on the top and bottom faces, the convergence of parametric lines signifies singularity at the poles (figure 17(b)). On the other hand, on projecting a parameterized cube on a sphere, we observe a singularity free map (figure 17(c)).

REPRESENTING ROM FOR COUPLED JOINTS ON A CUBE - The ranges of motion of 2DOF and 3DOF joints are generally not independent of each other. Projecting spherical surfaces on parametric cubes can be used as an effective method to parameterize ranges of motion of arbitrary spherical joints and other coupled joints for digital human modeling. The joint ranges of motion are generally represented as spherical polygons by using certain key orientations [5] [6] or as spherical triangulated surfaces [12] [13]. Such polygons and surfaces can be projected on a cube preserving the linearity of segments and can be used to classify directions by converting the problem to point-classification problem as discussed in the earlier section. Even if the ranges of motion are represented using control polygons for interpolating the range into smooth curves (rational B-splines), the proposed method can be used by projecting the control polygon to the cube and constructing the curve on the cube itself. This advantage comes by the virtue of projective invariance of smooth curves. This section shows how the projection of spherical surfaces is implemented by taking projecting each triangle individually on the cube. Figure 18 shows the surface of the earth and its projection on the cube. Similarly any triangulated surface representing ranges of motion can be treated to obtain the parametric space and carry out point classification. It is observed that the use of spherical parameterization of a triangulated surface (the surface of earth in this case) is distorted to a considerable extent (figure 18(c)). It is to be noted that even when the surface taken in this experiment is in itself spherical, the parametric space is not obvious to speculate or in other words is distorted considerably when opened up. On the other hand, in case of a cube, what is seen on the faces of the cube is itself the parametric space when looked at face by face. The distortion of the shadow of the earth's surface is comparable to those using existing methods of map-projections [14].

REPRESENTING TWIST OF SPHERICAL JOINTS - The current literature shows how the ranges of motion of coupled joints and arbitrary spherical joints can be represented using sphere as object of projection [5][6]. Liu and Prakash [11] and Lee [15], use the quaternion approach for solving the problem of representation. Due to the non-linearity of spherical parametric space, the parameterization of continuous and smooth curves is computationally expensive. Further, using the conventional axis-angle and Euler-angle approaches does not support representation of the range of twist associated with spherical joints. Baerlocher and Boulic [6] has shown swivel-twist representation by using axis-angle approach but the problems associated with singularities still persist as this approach converts Euler-angles to axis-angle which inherently has singularities [16].

To solve the problem of representing twist on a cube we propose that once we get a set of directions in space as a closed planar curve or a planar polygon, the range of twist at any direction within the given set can be represented as two lines with length equal to the maximum and minimum values of permitted twist along and anti-parallel to the normal of the plane at the point of concern. In general, if the set of directions form a non-planar surface, the same approach can be applied by drawing the lines of twist according to the normal at a point (representing the direction) of concern. Given a set of directions and their respective ranges twist, the twist can be projected at every corresponding projection of a direction on the cube as shown in figure 20. The proposed idea has been implemented by taking a nominal plane containing the polygon (for ranges of swivel and swing) and a surface of rotation above and below this plane (figure 21). It is desirable to note here that this could be computationally too expensive if done using spherical parameterization.

CONCLUSION

The present work shows, how the representation of direction can be achieved by using multi-planar domains rather than a single spherical domain. In particular, this work focuses on a special multi-planar domain, "cube". The choice of position and size of the cube makes the parametric identification of a direction on the cube easy. Several characteristics of the projection of a line in space on the cube have been identified. The results obtained from projecting triangles in various configurations on the cube and the sphere demonstrates that the parametric space of the cube not only is devoid of singularity, but also preserves the linearity of the objects projected unlike its spherical counterpart which is highly nonlinear.

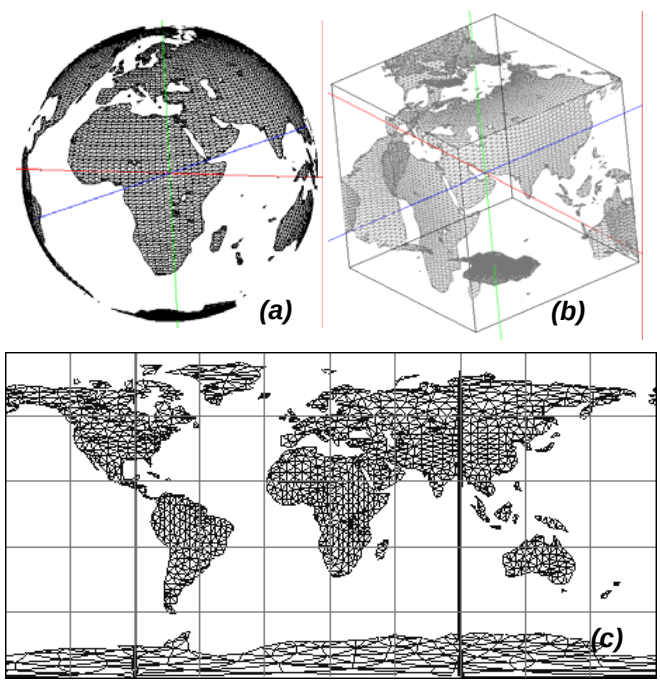

Figure 18: (a) Surface of earth on sphere
(b) Surface of earth on cube
(c) Earth in spherical parametric space

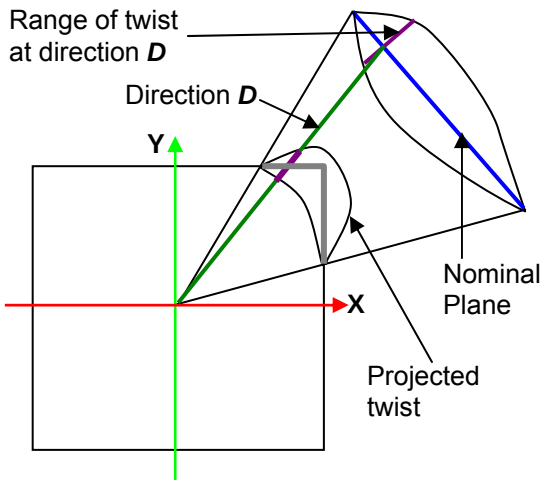

Fig 20: Schematic for projecting twist on a cube

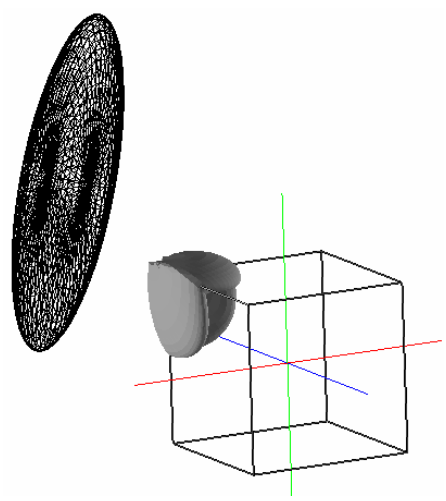

Fig 21: Representing twist on a cube

The method thus is computationally efficient and representationally simple. In the proposed scheme, the point-in-a-spherical-polygon problem reduces to a point-in-a-planar-polygon problem; which is well addressed in literature. The method thus transforms any spherical polygon into a set of plane polygons enabling simplified representation and usage of joint ranges of motion. Extension of the scheme to include the representation of the range of twist of spherical joints has also been illustrated.

The proposed method is easily extendable to generate Gauss maps, visibility maps and similar class of problems.

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