Voronoi Spaghetti & VoroNoodles: Topologically Interlocked, Space-Filling, Corrugated & Congruent Tiles

Cassie Mullins
cmullins7@email.tamu.edu
Department of Visualization, CoA
Texas A&M University
College Station, Texas, USA

Matthew Ebert
matt_ebert@tamu.edu
Department of Mechanical Engineering, CoE
Texas A&M University
College Station, Texas, USA

Ergun Akleman
ergun.akleman@gmail.com
Visual Computing & Computational Media, SPVFA
Computer Science and Engineering, CoE
Texas A&M University
College Station, Texas, USA

Vinayak Krishnamurthy
vinayak@tamu.edu
Department of Mechanical Engineering, CoE,
Computer Science and Engineering, CoE
Texas A&M University
College Station, Texas, USA

(a) *Ruled surface.* **(b)** *A printed single VoroNoodle.* **(c)** *Assembly of two printed VoroNoodles.* **(d)** *Assembly of three printed VoroNoodles.* **(e)** 2×2 *Assembly of printed VoroNoodles.* **(f)** 2×2 *Assembly of printed VoroNoodles.*

Figure 1: An example of congruent VoroNoodles that provide strong topological interlocking. Each layer is translated into a parametric equation in the form of $x = cos(3z)$ and $y = sin(3z)$ to produce a helix. To obtain the helical ruled surface shown in 1a, we rotated and scaled a 2D vector along this helix. Translated versions of these ruled surfaces are used as Voronoi sites to obtain Corrugated Bricks, which we call VoroNoodles.

ABSTRACT

In this work, we introduce an approach to model topologically interlocked corrugated bricks that can be assembled in a water-tight manner (space-filling) to design a variety of spatial structures. Our approach takes inspiration from recently developed methods that utilize Voronoi tessellation of spatial domains by using symmetrically arranged Voronoi sites. However, in contrast to these existing methods, we focus our attention on Voronoi sites modeled using helical trajectories, which can provide corrugation and better interlocking. For symmetries, we only use affine transformations based on the Bravais lattice to avoid self-intersections. This methodology naturally results in structures that are both space-filling (owing to Voronoi tessellation) as well as interlocking by corrugation (owing to helical trajectories). The resulting shapes of the bricks appear to be similar to a variety of pasta noodles, thereby inspiring the names, *Voronoi Spaghetti* and *VoroNoodles*.

CCS CONCEPTS

• **Computing methodologies** → **Mesh models**; **Volumetric models**.

KEYWORDS

Space Filling Tiles, Voronoi Decomposition, Delaunay Diagrams, Lofting

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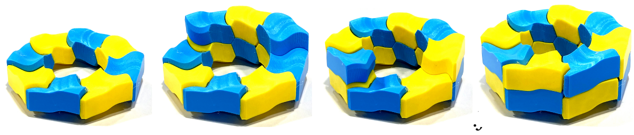
1 INTRODUCTION

There have been several recent works to obtain 2.5D space-filling tiles by using layer-by-layer Voronoi decomposition [Akleman et al. 2020; Subramanian et al. 2019]. We observe that these works do not guarantee topological interlocking, which is defined as assemblies



(a) 8 *pieces.* **(b)** 11 *pieces.* **(c)** 14 *pieces.* **(d)** 16 *pieces.*

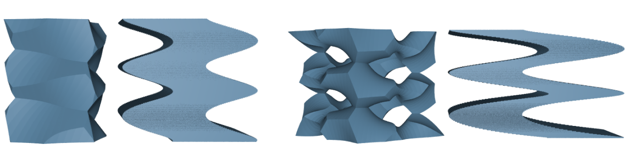
Figure 2: Congruent VoroNoodles on a cylindrical domain. These show cylindrical assemblies of multi pieces of printed VoroNoodles.

of tiles that stay in place purely owing to the kinematic constraints imposed by their neighboring tiles under a peripheral force [Estrin et al. 2019, 2021]). In this work, we demonstrate that it is possible to guarantee topological interlocking through corrugation. Our approach is based on decomposing a given volume using helical ruled surfaces as Voronoi sites that populate volume using Bravais lattices. The shapes are still obtained by layer-by-layer Voronoi decomposition (See Figure 3), which is also critical to obtain strong corrugation. Layer-by-layer Voronoi decomposition also guarantees obtaining genus-0 surfaces and the resulting tiles can always be assembled. An important property of this approach is that it provides access to a large design space to produce topologically interlocking tessellations. Using this approach, we introduce two unique varieties, namely, *Voronoi Spaghetti* (shapes that have the same cross-section akin to a swept volume — Figure 4) and *VoroNoodles* (shapes that have varying cross-sections akin to lofted volumes — Figure 5) This flexibility in design space is quite helpful to systematically search for new tessellated forms.

1.1 Basis & Rationale

Our general idea is based on using screw shapes as Voronoi cells. To picture mentally how we get corrugated brick regions imagine that N number of screws screwed into a thick architectural shell or slab. We partition this shell into N number of brick regions where region i consists of all points closest to screw number i. This operation naturally creates desired corrugated boundaries between the brick regions. It should be clear that shapes and placements of screws play important roles to obtain interesting corrugated bricks.

The first requirement is that no two screws intersect with each other. This is not physically possible but our virtual screws can potentially intersect with each other if we randomly place them. To obtain a nice decomposition of the space, we also need screws to be placed *as-uniformly-as-possible* inside of the shell. This means no two screws have to be too close to each other and there should not be any large region left without a screw. Boris Delaunay introduced the concept of "Delone Set" to formally describe such *as-uniform-as-possible* placements [Delaunay and Sandakova 1961; Delone 2005; Schmitt 2016].



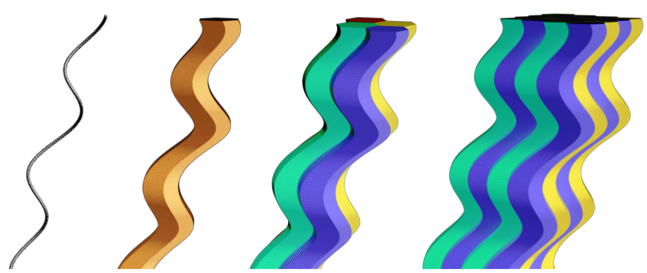
(a) *3D Voronoi at the left does not provide strong corrugation.* **(b)** *3D Voronoi at the left is not genus-0.*

Figure 3: Comparison of 3D vs. 2.5D (i.e. layer-by-layer) Voronoi decomposition [Subramanian et al. 2019] using the same data sets. Note that 3D Voronoi can even produce high-genus tiles that may not necessarily be assembled. 2.5D provides both strong corrugations and guarantees to create the genus-0 surfaces.

Delaunay introduced two concepts, (1) uniformly discrete and (2) relatively dense, to define *as-uniform-as-possible* placements of points. Now, let S denote a set of points in n-dimensional Euclidean space, $\Re^n$, the S is called a Delone set if it is both uniformly discrete and relatively dense [Delone et al. 1970]. Let $r_1 > r_0$ denote two positive numbers: then (1) the point set S is uniformly discrete if each ball of radius r_0 contains at most one point in S; (2) the point set S is relatively dense if and every ball of radius r_1 contains at least one point of S [Delone et al. 1976]. If we use the points in $S \in \Re^3$ as Voronoi sites, we can obtain a "*nice*" tessellation of space that contains. This partition is nice in the sense that the resulting Voronoi cells are similarly-sized convex polyhedra.

Even though the notions *uniformly discrete* and *relatively dense* are defined and can be intuitively understood for points only, they are still useful if the sizes of higher-dimensional Voronoi sites are much smaller than r_0, where we can define the size of the object as the radius of a bounding sphere. Our problem, however, requires an alternative approach. If the screws are away from each other, they will really act like points, and boundaries between Voronoi regions will not have many corrugations. In order to produce corrugated boundaries, we need to place screws as-close-as-possible. In this case, the sizes and shapes of the screws play an important role to obtain uniformly discrete and relatively dense distributions.

The representations of screws can be obtained by curves and ruled surfaces that are defined in the parametric domain $(x, y, z) \in$



(a) *Single curve.* **(b)** *A single Voronoi Spagetti.* **(c)** *2 × 2 assembly of Voronoi Spaghetti.* **(d)** *4 × 4 assembly of Voronoi Spaghetti.*

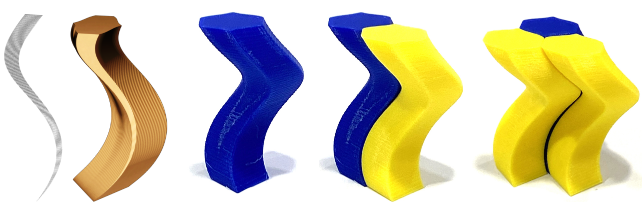
Figure 4: An example of Voronoi Spaghetti as an extruded 2-Honeycomb. Since the 2-Honeycomb is exactly the same in each layer, each congruent tile is exactly the same in each layer. Since this is obtained by a single curve, the resulting congruent tiles in each layer are convex polygons. In this particular case. they are all regular hexagons.Each layer is translated using this parametric equation: $x = 2cos(2\pi t) + cos(2\pi 5t)$, $y = 2sin(2\pi t) + sin(2\pi 5t)$, and $z = 4t$. We used $|v_0| << 1$ and $|v_1| << 1$ to obtain long and skinny spaghetti look.

$[0,1]^3$. They are defined by trigonometric functions to control the frequency of corrugation intuitively. The intersection of screw shapes with any z constant planar layer is guaranteed to be a point or a line. With points and lines, it is easy to obtain uniformly discrete and relatively dense distribution in each layer. This further allows one to pack long curves or ruled surfaces as close as possible to each other without intersection.

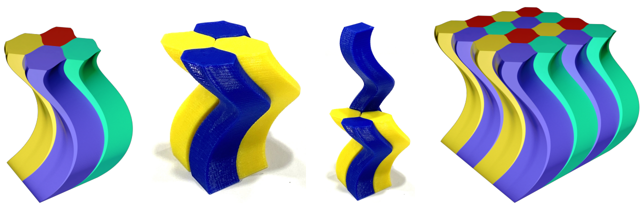
For placement and arrangement of the Voronoi sites, we employ a strategy developed by Auguste Bravais [Bravais 1850] that allows populating the system by using only translation. Using Bravais lattices, we guarantee to obtain *symmetric* Delone sets in the parametric domain [Delaunay and Sandakova 1961]. Because of symmetry, the Voronoi decomposition of a parametric domain results in a Voronoi tessellation that consists of congruent bricks. The congruence of the corrugated bricks is preserved for some bijective mapping such as scale, shear, or cylindrical transformations (See Figure 2 for a cylindrical assembly of congruent corrugated bricks). However, for most, it is not preserved. The resulting corrugated bricks appear similar but will not exactly be the same.

1.2 Challenges & Approach

Even though there exist previous works that utilize similar approaches to obtain topologically interlocking tiles such as Delaunay Lofts and Generalized Abeille Tiles [Akleman et al. 2020; Subramanian et al. 2019] they do not provide corrugated boundaries. Providing topological interlocking by helical ruled surfaces as Voronoi sites leads to intuitive and generalizable design mechanisms to obtain desired corrugated boundaries. For instance, an intuitive way of defining such a ruled surface is rotating and scaling a line while translating its center. To demonstrate this intuitive approach, we choose a concrete class of helical trajectories wherein we map the motion of each Voronoi site based on the transformation matrices induced by the helix. Our results show that the resulting congruent shapes coming from helical trajectories demonstrate strong potential for topological interlocking assemblies by providing corrugated boundaries.



(a) *Ruled Surface.* **(b)** *Rendered single VoroNoodle.* **(c)** *Printed single VoroNoodle.* **(d)** *Assembly of two printed VoroNoodles.* **(e)** *Assembly of three printed VoroNoodles.*



(f) *Rendered assembly of four VoroNoodles.* **(g)** *Assembly of four printed VoroNoodles.* **(h)** *Five VoroNoodles.* **(i)** *Ruled Surface that is used as Voronoi site.*

Figure 5: An example of VoroNoodles with a different 2-Honeycomb in each layer. This is obtained by using a ruled surface as a Voronoi site, created by scaling a line. Each layer is translated using a $\mathrm{p}(t)$ in the form of $x = cos(2\pi t)$, $y = sin(2\pi t)$, and $z = t$ to create a helix. For $\vec{v}(t)$, we scaled and rotated a 2D vector.

2 CONCEPTUAL FRAMEWORK

To present our approach we first use an infinite array of a single curve as Voronoi sites. Such curves create spaghetti-like extruded structures with the same convex polygon type cross-sections along the z direction as shown in Figure 4d. We then extend these curves to ruled surfaces to obtain VoroNoodles, in which every layer can have a different cross-section. Moreover, the shapes of cross-sections can be non-convex and boundary edges do not have to be straight. Therefore, *VoroNoodle* boundaries can be more complex than ruled surfaces.

2.1 Designing Voronoi Spaghetti

Consider a continuous curve $p : [0,1] \to \Re^3$ defined as

$$\mathbf{p}(t) = (F_x(t), F_y(t), at)$$

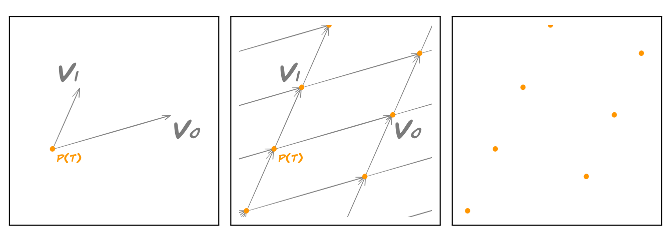
where $t \in [0,1]$ and a is any positive real number that is used to scale VoroNoodles along z direction. The functions $F_x : [0,1] \to \Re$ and $F_y : [0,1] \to \Re$ can be any function as long as they are continuous. They do not need to be C^1 or C^2.

Now, assume that a Bravais lattice is defined by two linearly independent (but not necessarily mutually perpendicular) translation vectors v_0 and v_1 that can span the $x-y$ vector space given by the two orthonormal vectors $(1,0,0)$ and $(0,1,0)$ We use this Bravais Lattice to produce an infinite array of curves by adding the vector $n_0\vec{v}_0 + n_1\vec{v}_1$ to the curve as

$$\mathbf{p}(t) + n_0\vec{v}_0 + n_1\vec{v}_1$$

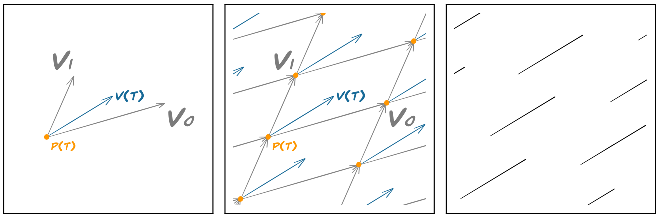
where the n_0 and n_1 are any integers to span the lattice. Regardless of how we choose functions F_x and F_y and the Bravais vectors v_0 and v_1,

$$\mathbf{p}(0) + n_0\vec{v}_0 + n_1\vec{v}_1$$



(a) *Initial point and Bravais vector in layer t.* **(b)** *Bravais translation of the point in layer t.* **(c)** *Delone set of points in layer t.*

Figure 6: The line in each layer will stay inside of the parallelogram defined by the Bravais vectors $\vec{v}_0$ and $\vec{v}_1$.



(a) *Initial line in layer t.* **(b)** *Bravais translation of lines in layer t.* **(c)** *Delone set of lines in layer t.*

Figure 7: The line in each layer will stay inside of the parallelogram defined by the Bravais vectors $\vec{v}_0$ and $\vec{v}_1$.

is guaranteed to be a symmetric Delone set. Its symmetry is uniquely defined by the vectors v_0 and v_1 as shown in Figure 6. For any given t we obtain only translated versions of the same symmetric Delone set as in Figure 6c. Therefore, this formulation can be considered as an extrusion of symmetric Delone set along the curve $\mathbf{p}(t)$. The type of extrusion is uniquely defined by the choice of functions F_x and F_y.

If we apply 2D Voronoi in each layer, we obtain the same 2-Honeycomb in each layer. As a result, this will give us an extrusion of a 2-Honeycomb defined by Bravais lattice along with the curve $\mathbf{p}(t)$ or $\mathbf{p}(z)$. In other words, these congruent tiles consist of the same convex and congruent polygons that are stacked on top of each. These shapes will really look like long and skinny spaghetti if $|\vec{v}_0| << 1$ and $|\vec{v}_1| << 1$ as shown in Figure 4. The shape of convex and congruent polygons depends on only relative orientations and lengths of Bravais vectors $\vec{v}_0$ and $\vec{v}_1$. The shape can be a square, a rectangle, a regular hexagon, and a more general hexagon.

2.2 Designing VoroNoodles

Let a vector curve defined $[0, 1] \rightarrow [0, 1)^2$ defined as

$$\vec{v}(t) = G_x(t)\vec{v}_0 + G_y(t)\vec{v}_1$$

where $t \in [0, 1]$ and the functions $G_x : [0, 1] \rightarrow [0, 1)$ and $G_y : [0, 1] \rightarrow [0, 1)$ are continuous. Now, consider the ruled surface that is defined by the following equation:

$$\mathbf{P}(t, u) = \mathbf{p}(t)(1 - u) + (\mathbf{p}(t) + \vec{v}(t))u,$$

where $u \in [0, 1]$. This ruled surface consists of lines in every $z =$ constant layer. Further, consider an infinite array of these ruled surfaces as Bravais lattice by adding the vector $n_0\vec{v}_0 + n_1\vec{v}_1$ to the surface as

$$\mathbf{P}(t, u) + n_0\vec{v}_0 + n_1\vec{v}_1$$

where the n_0 and n_1 are any integers to span the lattice. In these case, regardless of how we choose the vector function $\vec{v}(t)$, the infinite array of ruled surfaces never intersect with each other since we choose the two components of the vector function less than $|\vec{v}_0|$ and $|\vec{v}_1|$ respectively. Therefore, the line segment will always stay in the parallelogram defined by the Bravais vectors $\vec{v}_0$ and $\vec{v}_1$ (See Figure 7).

This property is useful since we can formally guarantee each layer of an infinite array of lines to be a symmetric Delone set [Delone et al. 1976; Dolbilin et al. 2021]. This is based on Dolbilin's result that demonstrates if n-number of symmetric Delone sets represent the same crystallographic (in 2D wallpaper) group their union also represents the same wallpaper group with a crystallographic orbit of n-number points. Using this result, it is possible to to be arbitrarily close to any given higher-dimensional shape such as a planar curve, which can result of 2D crystals. Now, assume that we replace points in a symmetric Delone set with lines. Their Voronoi decomposition allows the creation of cell-transitive 2-honeycombs with congruent planar shapes that can have curved edges.

This is not the only method to obtain 2D Delone sets of lines. For instance, we can start with any two curves $\mathbf{p}_0(t)$ and $\mathbf{p}_1(t)$; and we can obtain any ruled surface as

$$\mathbf{P}(t, u) = \mathbf{p}_0(t)(1 - u) + \mathbf{p}_1(t)u.$$

For this ruled surface we can always construct two Bravais vectors using two given direction $\vec{n}_0$, and $\vec{n}_1$ as $\vec{v}_0 = a_0\vec{n}_0$, and $\vec{v}_1 = a_1\vec{n}_1$ by choosing a_0 and a_1 large enough to avoid intersection regardless of how initial curves are chosen. Note that the choice of $\vec{n}_0$, and $\vec{n}_1$ and a_0 and a_1 are not unique for any given sets of two curves that define a ruled surface.

In conclusion, this particular approach provides a large design space to construct topologically interlocking assemblies by guaranteeing to obtain different symmetric Delone sets in every layer for any set of curves and Bravais vectors. If the lines are not the same for each layer, the Voronoi partition of each layer produces different congruent planar shapes as shown in Figure 5 by providing additional local constraints.

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